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Memory Loss Process and Non-Gibbsian Equilibrium Solutions of Master Equations

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The phonon dynamics of a harmonic oscillator coupled to a steady reservoir is studied. In the Markovian limit, the equilibrium is reached through a progressive loss of memory process which involves the moments of the initial distribution. The relationship to the non-Markovian equations of motion and its resolvent poles is settled. As a particular model of the coupling mechanism is adopted, the possibility of non-Gibbsian equilibrium distribution arises, which is analyzed focusing upon the dependence of various parameters of the system on an effective equilibrium temperature.

KEY WORDS: Markovian limit; progressive loss of memory; non-Markovian analysis; non-Gibbsian equilibrium distribution.

1. INTRODUCTION

The features of quantal Brownian motion can be investigated through the quantization of a classical system that generally leads to a quantum Langevin-type equation.[1–5] The comprehension of the phenomenon may be enhanced if we consider two interacting subsystems neither of which nor the whole system necessarily possesses *a priori* a classical analog. Regarding this matter, a recent approach to the description of damped collective motion of finite quantal systems such as nuclei has been proposed[6] and various applications have been given.[7–16] The model involved in these works consists of a harmonic oscillator whose phonons interact with fermions which constitute a heat reservoir; we have recently investigated a more accurate modified version of the original model that takes into account non-Markovian effects in certain configurations at zero

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temperature.[17,18] In the present work we further develop the approach for the general case of nonvanishing temperature.

The quantal Brownian motion of a harmonic oscillator also constitutes a problem of interest in the context of quantum optics.[19-21] In this framework, the damped oscillator has been generally treated by means of Glauber's P quasiprobability distribution,[22] which, in such a case, moves according to an analytically solvable Fokker–Planck equation.[23-25] However, the solution through Glauber's P function involves certain disadvantages, since it is unsuitable for describing full quantum configurations, such as:

1. The relaxation of a pure n-phonon state, which is the most interesting initial condition when one focuses upon damped collective motion of finite quantal systems such as nuclei.[6-9] In fact, it is well known[22] that a pure n-phonon state cannot be represented by a well-behaved P function (the P representation corresponding to a density matrix $|n\rangle\langle n|$ contains derivatives of delta functions up to order $2n$).[26]

2. The evolution toward a configuration with vanishing equilibrium temperature. In fact, in such a situation it can be shown that both the Fokker–Planck equation and its solutions collapse.[23] This collapsing behavior arises from the fact that all the eigenvectors of the master equation have, at zero temperature, an infinite number of vanishing components[18] and therefore they do not possess a well-behaved associated P function.[22]

The above drawbacks of the P solution led us to explore the direct solutions of the master equation.

This paper is organized as follows: first we solve the master equations for the phonon populations in the Markovian limit (Sections 2 and 3). Since the classical techniques of solution of master equations such as the Kirchhoff[27] or continued-fraction[25] methods are not useful in the present case, we use the characteristic function associated with the probability distribution governed by the master equation. This procedure allows us to find the complete dynamical solution, which is analyzed in Section 3, giving rise to an interesting process with progressive loss of memory. In Section 4 we show that the non-Markovian equations can be easily solved by means of the Markovian solutions. The phonon dynamics in the non-Markovian regime is examined in Section 5 for a particular model of the system-to-reservoir coupling and a non-Gibbsian behavior is observed,[1-3,17,18] which we study, focusing upon the dependence of various parameters of the model on an effective equilibrium temperature. The main results are summarized in Section 6.

2. THE SPECTRAL PROBLEM OF THE MARKOVIAN MASTER EQUATION

In this section, we illustrate the use of characteristic functions to extract the eigenvalues and eigenvectors of a statistical generator of evolution. The starting point is the Markovian master equation for the phonon populations ρ_n that corresponds to an oscillator immersed in a heat bath.[23-25] This equation reads, for $n \geqslant 0$,

$$\dot{\rho}_n = nW_- \rho_{n-1} - [nW_+ + (n+1) W_-] \rho_n + (n+1) W_+ \rho_{n+1} \tag{2.1}$$

with the microscopic transition rates $W_\pm$, whose structure depends on the particular model for the heat bath and the interaction.

Now let us consider the characteristic function for the probability distribution $\rho_n(t)$,

$$\psi(x, t) = \sum_{n \geqslant 0} \rho_n(t) x^n \tag{2.2}$$

From Eq. (2.1), we can easily obtain an equation for this function, namely,

$$\frac{\partial \psi(x, t)}{\partial t} = W_+ (x-1) \left[\beta \psi(x, t) + (\beta x - 1) \frac{\partial \psi(x, t)}{\partial x} \right] \tag{2.3}$$

where we have defined the parameter β

$$\beta = W_- / W_+ \tag{2.4}$$

We investigate the stationary solutions of Eq. (2.3), namely

$$\psi^{(k)}(x, t) = \chi^{(k)}(x) \, e^{k(W_- - W_+)t} \tag{2.5}$$

where k is a number to be determined. Separation of variables in (2.3) leads us to the eigencharacteristics,

$$\chi^{(k)}(x) = \frac{\rho_0^{(k)}}{1 - \beta x} \left(\frac{1 - x}{1 - \beta x} \right)^k \tag{2.6}$$

We can now find the solution of the spectral problem of the master equation (2.1) by means of Eqs. (2.5) and (2.6). In fact, the stationary ansatz (2.5) defines the set of eigenvalues

$$\lambda_k = k(W_- - W_+) \tag{2.7}$$

the corresponding eigenvectors being [cf. Eq. (2.2)]

$$\rho_n^{(k)} = \frac{1}{n!} \frac{d^n}{dx^n} \chi^{(k)}(x) \bigg|_{x=0} \tag{2.8}$$

Using expression (2.6), a straightforward calculation leads us to

$$\rho_n^{(k)}=\frac{\rho_0^{(k)}}{n!}\sum_{j=0}^{n}\beta^{n-j}(-1)^j\binom{n}{j}(k-j+n)(k-j+n-1)\cdots(k-j+1) \qquad (2.9)$$

for the nth component of the eigendistribution corresponding to the eigenvalue λ_k.

Similarly, the moment eigenvectors can be calculated from

$$(\overline{n^p})^{(k)}=\left(x\frac{d}{dx}\right)^p \chi^{(k)}(x)\bigg|_{x=1} \qquad (2.10)$$

Actually, these eigenvectors are the solutions of the spectral problem associated with the following equation of motion for the moments, which is derived from Eq. (2.1):

$$\dot{\overline{n^0}}=0 \qquad (2.11a)$$

$$\dot{\overline{n^1}}=W_-\overline{n^0}+(W_- - W_+)\overline{n^1} \qquad (2.11b)$$

$$\dot{\overline{n^p}}=W_-\overline{n^0}+\sum_{j=1}^{p-1}\left[\binom{p+1}{j}W_- +(-1)^{p+1-j}\binom{p}{j-1}W_+\right]\overline{n^j} + p(W_- - W_+)\overline{n^p}, \qquad p\geqslant 2 \qquad (2.11c)$$

From Eqs. (2.11) an easy algebraic calculation leads us to a spectrum of the analytical form (2.7). In addition, it becomes evident that the number k must be nonnegative integer (in Section 5 we discuss the conditions under which one has $W_- < W_+$, i.e., λ_k nonpositive). This spectrum was reported in ref. 23, from which one can verify after some calculations that the inclusion of nondiagonal elements of the density matrix to the dynamics adds all positive half-integers to the spectrum of eigenfrequencies. However, these frequencies do not appear in the dynamics when one focuses upon a diagonal initial density, such as a pure n-phonon state.

The calculation of the eigenmoments is lengthy and tedious; we do not consider the analytical form of the results of any special interest to be shown here, except for the property

$$(\overline{n^p})^{(k)}=0 \qquad \text{if} \quad p<k \qquad (2.12)$$

In particular, the "mass" $\overline{n^0}=\mathrm{Tr}\,\rho$ of a distribution ρ gives for the eigenvectors

$$(\overline{n^0})^{(k)}=\chi^{(k)}(1) \qquad (2.13)$$

Inspection of Eq. (2.6) demonstrates that all eigenvectors are traceless[27] except for the equilibrium distribution corresponding to $k=0$, which possesses the canonical structure with a "Boltzmann factor" equal to β in (2.4).[23,25] We will refer in Section 5 to the conditions under which one gets a Gibbsian distribution $\beta = \exp(-\hbar\Omega/T)$, with T the temperature of the heat bath, as well as the meaning of non-Gibbsian solutions, which has also been discussed in ref. 17.

3. DISTRIBUTION AND PHONON DYNAMICS

In the preceding section we calculated the eigenvalues and the eigenvectors of the matrix M associated with the master equation (2.1). Accordingly, the time evolution of the density matrix can then be expressed as

$$\rho_n(t) = \sum_{k \geqslant 0} A_k e^{\lambda_k t} \rho_n^{(k)} \tag{3.1}$$

with amplitudes A_k whose values can be obtained from the initial occupations $\rho_n(0)$ and the eigenvectors $\rho^{+(k)}$ of the adjoint master equation[27] as

$$A_k = \sum_{m \geqslant 0} \rho_m^{+(k)} \rho_m(0) \tag{3.2}$$

The calculation of these adjoint eigenvectors follows the same steps as in the preceding section, starting from the equation of motion of an adjoint characteristic function $\psi^+(x, t)$. One finally obtains the complete solution of the spectral problem as

$$\begin{Bmatrix} \rho_n^{(k)} \\ \rho_n^{+(k)} \end{Bmatrix} = (1-\beta)^{1/2} \sum_{j=0}^{\inf(k,n)} (-)^j \begin{Bmatrix} \beta^{n-j} \\ \beta^{k-j} \end{Bmatrix} \binom{n}{j} \binom{k+n-j}{n} \tag{3.3}$$

for $k = 0, 1, 2, \ldots$. One can verify that taking $\beta = 0$, these results coincide with those explicitly derived in ref. 18.

Using Eq. (3.3), we get

$$A_k = (1-\beta)^{1/2} \sum_{m \geqslant 0} \rho_m(0)\, P_k(m, \beta) \tag{3.4}$$

where $P_k(m, \beta)$ is a kth-degree polynomial in m parametrized by β. Expression (3.4) is then an important relationship indicating that the amplitudes A_k depend on the initial phonon configuration through the first $k+1$ moments of the initial distribution, i.e., from $\overline{n^0}(0)$ to $\overline{n^k}(0)$. Notice,

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however, that the amplitude A_0 corresponding to zero decay rate is independent of the initial condition, since

$$A_0 \rho_n^{(0)} = (1-\beta)\,\beta^n = (1-\beta)^{1/2}\,\rho_n^{(0)} \tag{3.5}$$

i.e., $A_0 = (1-\beta)^{1/2}$. The attractor of the dynamical system (2.1) is then the canonical distribution that is reached, for very long times, when the system has forgotten every detail of its initial configuration. Along the same line of reasoning, we may interpret the long-term time evolution where $\rho_n(t)$ in (3.1) can be approximated by

$$\rho_n(t) \approx \rho_n^{(0)} + A_1[\bar{n}(0)]\, e^{\lambda_1 t} \rho_n^{(1)} \tag{3.6}$$

as a stage where only memory of the phonon number is kept. Similarly, for medium-term time evolution, where we may express, for some finite s,

$$\begin{aligned} \rho_n(t) = \rho_n^{(0)} &+ A_1[\bar{n}(0)]\, e^{\lambda_1 t} \rho_n^{(1)} + A_2[\bar{n}(0), \overline{n^2}(0)]\, e^{\lambda_2 t} \rho_n^{(2)} \\ &+ \cdots + A_s[\bar{n}(0),..., \overline{n^s}(0)]\, e^{\lambda_s t} \rho_n^{(s)} \end{aligned} \tag{3.7}$$

the system only "remembers" the first s initial moments.

We can summarize the above observations and offer a full description of the loss-of-memory process that accompanies the time evolution; indeed, this loss of memory takes place as the system progressively forgets the information contained in the highest moments of the initial distribution. Equilibrium is reached when the oscillation eventually forgets its initial mean phonon number.

We can also study the phonon dynamics by means of the motion of the moments. In particular, the first two moments read

$$\bar{n}(t) = \frac{\beta}{1-\beta} + \left[\bar{n}(0) - \frac{\beta}{1-\beta}\right] e^{\lambda_1 t} \tag{3.8a}$$

$$\begin{aligned} \overline{n^2}(t) = \frac{\beta(\beta+1)}{(1-\beta)^2} &+ \frac{3\beta+1}{1-\beta}\left[\bar{n}(0) - \frac{\beta}{1-\beta}\right] e^{\lambda_1 t} \\ &+ \left[\overline{n^2}(0) - \frac{3\beta+1}{1-\beta}\,\bar{n}(0) + \frac{2\beta^2}{(1-\beta)^2}\right] e^{\lambda_2 t} \end{aligned} \tag{3.8b}$$

We notice that while the mean phonon number approaches its equilibrium value corresponding to the canonical distribution with a damping rate $|\lambda_1| = W_+ - W_-$, the dispersion $\overline{n^2}(t) - [\bar{n}(t)]^2$ admits as well a contribution from the frequency $|\lambda_2| = 2(W_+ - W_-)$. We also observe in Eqs. (3.8) that the coefficients that weight each decaying factor $e^{\lambda_k t}$ with $k = 1$ or 2 only depend upon the $k+1$ lowest initial moments, a fact that is consistent with our previous analysis of the amplitudes A_k.

The features of the phonon dynamics can then be generalized through the following statement: the time evolution of the pth moment $\overline{n^p}(t)$ is exclusively built up of the p lowest nonvanishing frequencies, while each weighting coefficient of a decaying factor with frequency $|\lambda_k|$ appearing in either the probability or the moment expansion only depends on the first $k+1$ initial moments.

4. THE NON-MARKOVIAN MASTER EQUATION

The non-Markovian master equation, whose Markovian limit is given in Eq. (2.1), exhibits the general form[18,20]

$$\dot{\rho}_n(t) = \int_0^t d\tau \, \{ W_+(\tau)[(n+1)\,\rho_{n+1}(t-\tau) - n\rho_n(t-\tau)] + W_-(\tau)[n\rho_{n-1}(t-\tau) - (n+1)\,\rho_n(t-\tau)] \} \tag{4.1}$$

In order to solve this integrodifferential system, it is convenient to perform a Laplace transformation, which leads to an algebraic system of the form

$$[\lambda I - M(\lambda)]\,\tilde{\rho}_\Omega(\lambda) = \hat{\rho}_\Omega(t=0) \tag{4.2}$$

In Eq. (4.2), $\tilde{\rho}_\Omega(\lambda)$ is the Laplace transform of the density vector $\hat{\rho}_\Omega(t)$ with components $\rho_n(t)$; I is the identity in Fock representation and the generator $M(\lambda)$ is the matrix

$$M(\lambda) = \begin{bmatrix} -W_- & W_+ & 0 & 0 & 0 \longrightarrow \\ W_- & -W_+ - 2W_- & 2W_+ & 0 & 0 \longrightarrow \\ 0 & 2W_- & -2W_+ - 3W_- & 3W_+ & 0 \longrightarrow \\ \downarrow & \ddots & \ddots & \ddots & \\ & nW_- & -nW_+ - (n+1)\,W_- & (n+1)\,W_+ & \\ & \searrow & \searrow & \searrow & \\ \downarrow & & & & \end{bmatrix} \tag{4.3}$$

with the microscopic transition rate-like functions

$$W_\pm = \tilde{W}_\pm(\lambda) = \int_0^\infty e^{-\lambda\tau} W_\pm(\tau)\,d\tau; \qquad \operatorname{Re}\lambda \geqslant 0 \tag{4.4}$$

Formally, the time evolution of the density vector $\hat{\rho}_\Omega(t)$ can be written from Eq. (4.2) as[2]

$$\hat{\rho}_\Omega(t) = \sum_{\lambda_k} \operatorname{Res}[R(\lambda), \lambda_k]\, e^{\lambda_k t} \hat{\rho}_\Omega(0) \tag{4.5}$$

[2] In Eq. (4.5) it is assumed that the singularity spectrum of the resolvent consists of single poles (cf. ref. 17).

where λ_k is the kth pole of the resolvent $R(\lambda) = [\lambda I - M(\lambda)]^{-1}$. Now, since we have completely solved the spectral problem of the matrix M and its adjoint in the preceding sections, we can set up an explicit form for the resolvent, namely

$$R(\lambda) = \sum_{k \geqslant 0} \frac{\hat{\rho}^{(k)}(\lambda)\, \hat{\rho}^{+(k)}(\lambda)}{\lambda + k[\tilde{W}_+(\lambda) - \tilde{W}_-(\lambda)]} \tag{4.6}$$

where $\hat{\rho}^{(k)}(\lambda)$ and $\hat{\rho}^{+(k)}(\lambda)$ are the eigenvectors, whose components can be obtained from Eq. (3.3) after the formal replacement of the parameter β by the function

$$\beta(\lambda) = \tilde{W}_-(\lambda)/\tilde{W}_+(\lambda) \tag{4.7}$$

Assuming analicity for $\beta(\lambda)$ (see Section 5) and hence the same for the eigenvectors $\hat{\rho}^{(k)}(\lambda)$ and $\hat{\rho}^{+(k)}(\lambda)$, then Eqs. (4.5) and (4.6) give us the density vector as

$$\begin{aligned} \hat{\rho}_\Omega(t) = \sum_{\substack{k \geqslant 0 \\ i}} [\exp(\lambda_{k_i} t)] \operatorname{Res}\left[\frac{1}{\lambda + k[\tilde{W}_+(\lambda) - \tilde{W}_-(\lambda)]}, \lambda_{k_i}\right] \\ \times [\hat{\rho}^{+(k)}(\lambda_{k_i}) \cdot \hat{\rho}_\Omega(t=0)]\, \hat{\rho}^{(k)}(\lambda_{k_i}) \end{aligned} \tag{4.8}$$

In this expression, λ_{k_i} denotes the ith root of the secular equation [cf. Eq. (2.7)]

$$\lambda + k[\tilde{W}_+(\lambda) - \tilde{W}_-(\lambda)] = 0 \tag{4.9}$$

while the dot in the second square bracket indicates the standard scalar product.

Equation (4.8) then represents the solution of the non-Markovian system (4.1) with given initial conditions $\hat{\rho}_\Omega(t=0)$.

5. PHONON DYNAMICS IN THE NON-MARKOVIAN REGIME

An investigation of the non-Markovian dynamics contained in Eq. (4.8) requires the study of the functions $w(\lambda) = \tilde{W}_+(\lambda) - \tilde{W}_-(\lambda)$ [Eq. (4.9)] and $\beta(\lambda)$ [Eq. (4.7)]. For this sake, we will adopt a definite model[6–18] that considers a particle–phonon interaction coupling the oscillator to a fermionic reservoir equilibrated at a temperature T. Furthermore, we assume the coupling to be inelastic with a finite duration τ_{col} in order to account for unobserved degrees of freedom which might be interacting as well with the vibrating mode. The consequences of such an assumption for the spectral problem have been discussed at length in refs. 17 and 18.

In such a model and under the Born approximation,[28] the non-Markovian transition kernels in Eq. (4.1) adopt the form

$$W_{\pm}(\tau) = (2g^2/\hbar^2) \sum_{\alpha\mu} |\lambda_{\alpha\mu}|^2 \, e^{-\gamma\tau} \cos(\omega_{\alpha\mu} - \Omega)\,\tau \begin{Bmatrix} \rho_\mu(1-\rho_\alpha) \\ \rho_\alpha(1-\rho_\mu) \end{Bmatrix} \tag{5.1}$$

which give the microscopic transition rate-like functions through Eq. (4.4):

$$\tilde{W}_{\pm}(\lambda) = (2g^2/\hbar^2) \sum_{\alpha\mu} |\lambda_{\alpha\mu}|^2 \frac{\gamma + \lambda}{(\gamma+\lambda)^2 + (\Omega - \omega_{\alpha\mu})^2} \begin{Bmatrix} \rho_\mu(1-\rho_\alpha) \\ \rho_\alpha(1-\rho_\mu) \end{Bmatrix} \tag{5.2}$$

The Markovian limit then arises from (5.2) upon taking $\lambda = 0$. As usual in this type of model, the transition probabilities are expressed in terms of the equilibrium Fermi distribution at temperature T, with Fermi energy ε_F,

$$\rho_A = \{1 + \exp[(\varepsilon_A - \varepsilon_F)/T]\}^{-1} \tag{5.3}$$

where A is either an α or μ single-fermion label. The transition rates also depend upon a spin-isospin degeneracy factor g and upon the coupling matrix elements $\lambda_{\alpha\mu}$ related to the creation or annihilation of a phonon with energy $\hbar\Omega$. We learn from Eq. (5.2) that the energy parameters in these functions are the inelasticy width $\gamma = \tau_{\text{col}}^{-1}$, the temperature T, the phonon frequency Ω, and the interaction matrix elements (IME) $\lambda_{\alpha\mu}$, since the single-fermion frequencies $\omega_{\alpha\mu}$ are usually related to the differences in the kinetic energy of the given orbitals.[6–17] It is now interesting to recall here (see ref. 17 for further details) how these parameters relate to each other in the construction of the Boltzmann factor β in Eq. (2.4). We know from previous work[6–18] that this factor coincides with $\exp(-\hbar\Omega/T)$ only in case the particle–phonon collisions are perfectly elastic, i.e., the inelasticity spread γ vanishes, since if this happens, the condition $\omega_{\alpha\mu} = \Omega$ gives rise to $W_+ = [\exp(\hbar\Omega/T)]\, W_-$. Accordingly, as we permit inelastic collisions to occur, we are making room in our description for some unspecified degrees of freedom whose role is to draw energy out of the collision vertex, where only particle and phonon labels are known. This fact provokes some uncertainty regarding the definition of an equilibrium temperature, since the extra degrees of freedom may not be equilibrated with the fermion bath. The modified Boltzmann factor β then contains the two possible manifestations of energy spread: the fermion temperature T and the inelaticity width γ. We could then introduce an effective equilibrium temperature T_{eff} as

$$T_{\text{eff}}(T, \gamma) = -\hbar\Omega/\ln\beta \tag{5.4}$$

and regard our overall system made up of the oscillator, the fermion reservoir, and the hindered coordinates as canonically equilibrated at a temperature T_{eff}.[17] A similar non-Gibbsian behavior has been observed by other authors[1-3] who studied a harmonic oscillator strongly coupled to a bath of harmonic oscillators. Exponentially decaying kernels similar to those displayed in Eq. (5.1) have been used in refs. 29 and 30 to simulate coupling to unspecified heat baths. In this context, γ^{-1} is interpreted as the fastest dissipative relaxation time introduced by the reservoirs.[29]

Kievsky and Hernandez[31] show a physical realization of the above hidden degrees of freedom, which are specifically attributed to a complete set of two particle–two hole fermionic configurations in a model for the damping process of a giant resonance in an axially symmetric nucleus.

Let us now consider the dependence of the equilibrium distribution on the IME. Such a distribution is given by

$$\rho_n^{(0)} = [1 - \beta(0)][\beta(0)]^n \tag{5.5}$$

where $\beta(0)$ arises from Eqs. (4.7) and (5.2). Now, if we consider the zero-temperature limit, we have

$$\beta(0) = \left(\sum_{\substack{\mu > F \\ \alpha < F}} \frac{|\lambda_{\alpha\mu}|^2}{\gamma^2 + (\omega_{\alpha\mu} - \Omega)^2} \right) \Bigg/ \left(\sum_{\substack{\mu < F \\ \alpha > F}} \frac{|\lambda_{\alpha\mu}|^2}{\gamma^2 + (\omega_{\alpha\mu} - \Omega)^2} \right) \tag{5.6}$$

where F denotes the Fermi level of the heat bath.

Equation (5.6) clearly exhibits those IMEs contributing to the balance dynamics in equilibrium at $T = 0$. Particular limits are the following:

1. Hole–particle IME ($\alpha < F$, $\mu > F$) negligible with respect to particle–hole IME ($\alpha > F$, $\mu < F$). In such a case, $\beta(0)$ approaches zero, causing the effective temperature in Eq. (5.4) to vanish as well. The equilibrium distribution is then the canonical one at zero temperature,

$$\rho_n^{(0)} = \delta_{n0} \tag{5.7}$$

This situation has been thoroughly investigated in refs. 17 and 18.

2. $\lambda_{\alpha\mu} = \text{const}$, independent of α, μ. Taking into account that

$$(\omega_{\mu\alpha} - \Omega)^2 > (\omega_{\alpha\mu} - \Omega)^2 \qquad \text{if} \quad \omega_{\alpha\mu} > 0 \tag{5.8}$$

we can see from Eq. (5.6) that $0 \leqslant \beta(0) \leqslant 1$, with

$$\lim_{\gamma \to \{^{\infty}_{0}} \beta(0) = \begin{cases} 1 \\ 0 \end{cases} \tag{5.9}$$

We can then establish the boundaries for the effective temperature,

$$T_{\rm eff}(T=\gamma=0)=0 \leqslant T_{\rm eff}(T=0,\gamma) \leqslant T_{\rm eff}(T=0,\gamma\to\infty)\to\infty \quad (5.10)$$

Expression (5.10) shows that the inelasticity spread may provoke a significant departure of the equilibrium distribution at zero temperature from the zero-temperature canonical one; indeed, the higher the effective temperature, the closer to a uniform pattern the equilibrium distribution lies.

A more general choice for the IME, which can be easily explored, is

$$\lambda_{\alpha\mu}=\begin{cases} c & \text{for} \quad \alpha>F,\ \mu<F \\ c' & \text{for} \quad \alpha<F,\ \mu>F \end{cases} \quad (5.11)$$

with a dimensionless parameter $A=c'/c$ giving the rate between hole–particle and particle–hole IME. Therefore, from Eq. (5.6) we get

$$\lim_{\gamma\to\infty} \beta(0)=A \quad (5.12)$$

and consequently

$$0 \leqslant T_{\rm eff}(T=0,\gamma) \leqslant -\hbar\Omega/\ln A \quad (5.13)$$

which means that the ratio A must range between zero and unity in the physically acceptable regime. In other words, hole–particle IMEs must be lower, on the average, than particle–hole ones, in order for the phonon dynamics expressed by the non-Markovian master equation to possess a physically meaningful attractor at zero temperature. It is not clear that such a restriction should persist for nonvanishing temperatures, since in such a case all IMEs contribute to the balance at thermal equilibrium.

We now turn to the analysis of the functions $w(\lambda)$ and $\beta(\lambda)$ parametrized by fermionic temperature T and inelasticity spread γ, keeping the phonon energy $\hbar\Omega$ as an energy unit whose value has been selected following previous work[17] as 13 (in arbitrary units). In refs. 17 and 18, the damping frequencies λ_{k_i} have been computed for the zero-temperature case with vanishing IME ratio A. We find that the shape of typical curves calculated in those previous works are not sensitively affected by increasing temperature and IME ratio. This is illustrated in Fig. 1a, where we plot $w(v)$ as a function of the real variable $v=-\lambda$ in the segment $0\leqslant v\leqslant\gamma$ for the cases $T=A=0$ and $0\leqslant T\leqslant 8$, $A=1$ ($\lambda_{\alpha\mu}=\text{const}$). In Fig. 1b we amplify the lower part of the plot drawn in Fig. 1a in order to appreciate the change in the slope of the lower curve. The effect of a varying temperature cannot be observed on the scales employed in these plots;

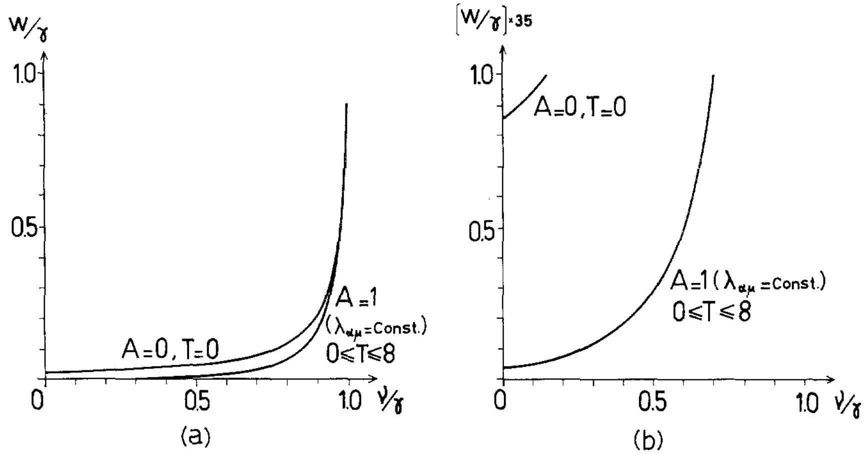

Fig. 1. (a) Plot of $w(\lambda) = W_+(\lambda) - W_-(\lambda)$ as a function of the real variable $\nu = -\lambda$ for the two extreme values of the IME ratio $A = 0$ and $A = 1$ (see text for explanation). The inelasticity spread is $\hbar\gamma = 100$, while the phonon energy is $\hbar\Omega = 13$ (arbitrary units). (b) Same as (a), on an amplified scale.

however, one can see from the data that increasing temperature always gives rise to increasing frequencies. This is shown in Fig. 2, where two curves corresponding to different temperatures have been qualitatively depicted.

From Fig. 1 one learns that the low-lying frequency, related to the lowest nonvanishing eigenvalue of the Markovian generator of the

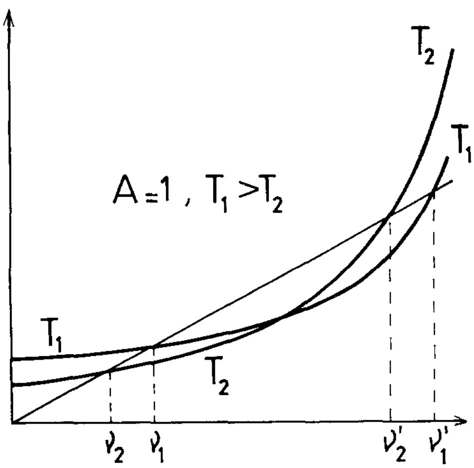

Fig. 2. The function $w(\nu)$ (with $\nu = -\lambda$) shown qualitatively for two values of fermionic temperature T_1 and T_2. The graphical solution of the secular equation (4.9) shows that increasing temperature gives rise to increasing decay frequencies.

motion,[17] significantly decreases as the strength parameter A grows. This decrease amounts to almost one order of magnitude from $A=0$ to $A=1$. Such behavior reflects the fact that increasing A ratios cause the downward- and upward-going transition rates $W_{\pm}(\lambda)$ to resemble each other, with the consequence that diffusive processes acquire a significant role in the dynamics, thus enlarging the characteristic decay time $\tau = |\lambda_{\min}|^{-1}$.

The situation is sensitively different when we consider the function $\beta(\lambda)$. Indeed, while $\beta(\lambda)$ vanishes for $T=A=0$,[17,18] it increases with A, as indicated in Fig. 3 for $A=1$ ($\lambda_{\alpha\mu}=\text{const}$) and various temperatures. We can observe that for any temperature, $\beta(\nu)$ is a decreasing function of $\nu=-\lambda$ in the range $0 \leqslant \nu \leqslant \gamma$. This is related to the behavior of the effective temperature, which is an increasing function of both T and γ, as shown in Fig. 4. In fact, we can see from Eqs. (5.2), (4.7), and (5.4) that for real λ equal to $-\nu$,

$$\beta(\nu, \gamma) = \beta(0, \gamma-\nu) = \exp[-\hbar\Omega/T_{\text{eff}}(T, \gamma-\nu)] \tag{5.14}$$

Therefore

$$\lim_{\nu \to \gamma} \beta(\nu, \gamma) = \beta(0, 0) = e^{-\hbar\Omega/T} \tag{5.15}$$

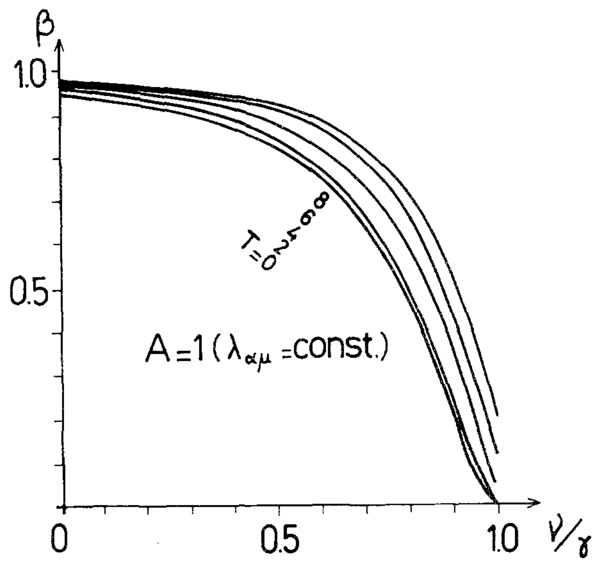

Fig. 3. Plot of $\beta(\lambda)=W_{-}(\lambda)/W_{+}(\lambda)$ as a function of the real variable $\nu=-\lambda$ for $A=1$ ($\lambda_{\alpha\mu}=\text{const}$) and various temperatures. The inelasticity spread is $\hbar\gamma=100$ and the phonon energy is $\hbar\Omega=13$ (arbitrary units).

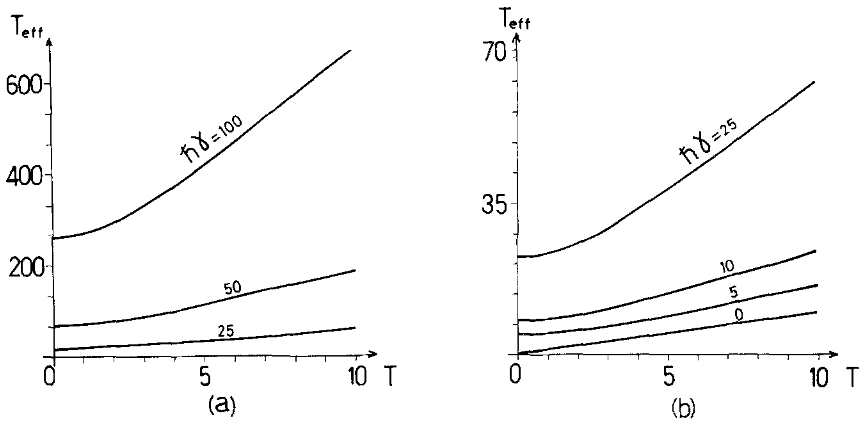

Fig. 4. The effective temperature $T_{\text{eff}}(T, \gamma)$ as a function of fermionic temperature T for given correlation width γ; two different drawings are displayed so as to illustrate the changing scale.

which is the right extreme ordinate of each curve in Fig. 3. On the other hand,

$$\lim_{\nu \to 0} \beta(\nu, \gamma) = \beta(0, \gamma) = \exp[-\hbar\Omega/T_{\text{eff}}(T, \gamma)] \tag{5.16}$$

is the left ordinate, from which we can read the effective temperature corresponding to the equilibrium situation for given T and γ.

The function $T_{\text{eff}}(T, \gamma)$ is plotted in Fig. 4 as a function of temperature for a given correlation width γ; two different drawings are displayed so as to illustrate the changing scale. It is clear that the two temperatures should approach each other as the energy spread parameter γ diminishes to zero, in order to permit the equilibrium distribution to reach the canonical limit with the Boltzmann factor (5.15), which is known to be the correct result for perfectly elastic collisions. One realizes that the effective temperature T_{eff} is always higher than the real one T, with increasing slope $\partial T_{\text{eff}}/\partial T$ as the inelasticity strength increases.

6. SUMMARY

In this work we have studied the phonon dynamics of a harmonic oscillator coupled to a steady reservoir. We have found that, in the Markovian limit, the oscillator reaches equilibrium through a particular loss-of-memory process: the system progressively forgets the information contained in the highest moments of the initial distribution. This trend is sustained until first the initial dispersion and finally the mean phonon number are forgotten when equilibrium is reached. We have analyzed a particular model including a fermion heat bath, which gives rise to a non-Gibbsian behavior of the equilibrium distribution, i.e., the oscillator is thermalized at an effective temperature which is generally different from the fermionic temperature. This effective temperature is strongly dependent on the form and strength of the interaction between fermions and phonons; in particular, one can show that some special choices of the interaction matrix elements lead to the nonexistence of a stationary solution. However, the effective temperature is generally a well-behaved, positive, increasing function of both parameters of energy spread, namely the fermion temperature T and the inelasticity width γ. The minimum of this function occurs for $T=0$ and is generally positive for nonvanishing inelasticity widths. A missing zero effective temperature provides a possible way out of the delicate problem[18] posed by the existence of complex frequencies that could give rise to oscillating probabilities around zero or unity (zero-temperature canonical distribution). Therefore, we must consider the analysis of the non-Markovian frequencies made in Section 5 as the starting point for a more complete description that will be presented in a future work.[32]

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